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# Limit of maximum entropy for the damped Jaynes–Cummings model

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## Abstract

Employing projection methods from kinetic theory, we study the non-Markovian quantum evolution of a two-level atom that is coupled to a single mode of the electromagnetic field. The interaction between the atom and field mode is described by the damped Jaynes–Cummings model. The general case is considered, in which dissipation is generated by both a photonic and an atomic reservoir of finite temperature. Only one special choice is made. The frequencies of the atom and field mode are in the same ratio as the temperatures of the atomic and the photonic reservoirs. Making use of Laplace transformation, we show that the atomic density matrix evolves to the state of maximum von Neumann entropy if the time, the cube of the initial electromagnetic energy density, the inverse of the photonic damping parameter and the inverse of the atomic damping parameter tend to infinity equally fast. We propose a large class of states from which the full density operator for the atom and field may start. This class includes entangled states. Expansion of the time-dependent exponential of the Laplace backtransform enables us to derive the limit of maximum entropy directly, without explicit evaluation of the atomic density matrix. We interchange limits and sums without proof, so our derivation is not entirely rigorous. Next, we remove the photonic reservoir. Then the damping process gets a sequential character. The field mode is assumed to start from a photon-number state. For the special choice of zero temperature and detuning we verify that the limit of maximum entropy survives the qualitative modification of the model. The only consequence is a slightly different scaling between the parameters that become large. Finally, we argue that our case study could be of value in finding out to what extent quantum dissipative processes obey the general principles of thermodynamics.

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## 1. Introduction

The Jaynes–Cummings model [1] with cavity damping offers an excellent opportunity to perform an in-depth examination of an open quantum system exhibiting non-Markovian dynamics. Owing to the well-known Rabi oscillations, which cause periodic energy exchange between the two-level atom and the privileged mode of the electromagnetic field, the atomic evolution possesses a quasi-reversible character for sufficiently small times. The main findings of the Jaynes–Cummings case study were published in paper I [2]. The cavity temperature and the detuning parameter were assumed to equal zero, because for that particular choice the model could be analytically solved [3]. Consequently, the behaviour of the atomic density matrix could be monitored with the help of modest numerical efforts.

Much attention was given to the atomic evolution in the regime of weak damping. This led to the numerical confirmation of a limit of maximum entropy, which was proposed in [3]. According to this result, the atomic density matrix was to converge to the so-called central state if the parameter for cavity damping became small, the initial energy of the electromagnetic field large, and the time large. Lying at the centre of atomic phase space, the central state maximizes the atomic von Neumann entropy. The evolution of this last quantity was plotted for suitable parameter choices, and indeed a rise to the maximum value was observed. It took place by oscillatory convergence to a plateau, the length of which was comparable to the lifetime of a photon. This perfectly stationary stage of the atomic evolution was followed by a more common exponential decay to zero, the value belonging to the atomic ground state.

In paper I, it was suggested that the limit of maximum entropy might be of some help in resolving the numerous open questions on the relation between non-relativistic quantum mechanics and the second law of thermodynamics [4]. As seen from a strict point of view, this statement was premature. Before starting any deeper investigations, one should rule out the unpleasant possibility that the entropy plots of I were the result of artefacts. These would stem from the particular choices that were made, for instance, the complete disregard of temperature effects. Obviously, conclusions of thermodynamic nature must be questioned, if they are based on microscopic derivations at zero temperature.

By the same token, one must critically regard those treatments that adopt a thermal state as the initial condition of the undamped Jaynes–Cummings model [5]. Preparation of this state requires interaction with a thermal reservoir for all negative times. The physical reasons why the reservoir is inactive at positive times remain obscure. If one chooses as the initial condition of the undamped Jaynes–Cummings model a direct product of an atomic state and a coherent state, then one generates the well-known evolution of collapse and revival of Rabi oscillations [6]. Numerous studies of this remarkable phenomenon have been performed with the aid of quantum entropy and related notions [2, 7].

The purpose of this paper is to put the thermodynamic relevance of paper I beyond all doubt. We are going to work with the general version of the damped Jaynes–Cummings model, for which both the atom and field are coupled to a reservoir of finite temperature. The only restriction is that the frequencies of the atom and field mode must be in the same ratio as the temperatures of the atomic and the photonic reservoirs. Again we derive a limit that maximizes the atomic von Neumann entropy. In contrast to I, we do not make special choices for the initial state of the density operator. Instead, we propose a large class of states that are allowed. Moreover, we verify that a change of the damping mechanism has no consequences for the existence of a limit of maximum entropy. This check is of utmost importance. If we want to give a thermodynamic twist to our findings, then surely they may not depend on such microscopic matters as the type of energy sink that is used.

To facilitate the reading of this paper, we present a short summary. In section 2, the master equation for the full density operator is converted to a  $c$ -number recursion in four dimensions. The latter can be put into algebraic form by means of a Laplace transformation. Consequently, the physics of the model is hidden in a set of contour integrals over meromorphic functions. As in [8], a completely general temperature transformation is carried out, which shifts one single pole to the origin of the complex plane. It guarantees that for large times the atom and field evolve to a thermal state. The final preparatory step consists of a matrix diagonalization, which makes the location of all poles explicit. Adopting the language of kinetic theory, we may say then that each pole generates a normal mode of the dynamics.

In section 3, we demonstrate that a separation of timescales occurs if the parameters for atomic and photonic damping become small. Projection techniques from kinetic theory [9] allow us to decompose the dynamics into a fast part, which relates to the Jaynes–Cummings interaction, and a slow part, which relates to the approach to thermal equilibrium. In the limit of weak damping these two subdynamics no longer depend on each other. The corresponding equations could very well be the starting point of further research, for instance, a precise study of the collapse and revival of Rabi oscillations [6] at finite temperature.

Ignoring this last option, we use the weak-damping description to establish, in section 4, the desired limit of maximum entropy. Through an expansion of the time-dependent exponential factor of the Laplace backtransform, we manage to construct a direct derivation, which does not demand explicit computation of the atomic density matrix. We interchange limits and sums without proof, so our derivation is not entirely rigorous. In contrast, in section 5 we do maintain mathematical rigour. Removing the photonic energy sink, we demonstrate that the limit of maximum entropy survives a qualitative modification of the damping process. Finally, in section 6 we argue that our case study indeed serves general purposes. The non-Markovian dynamics that has been uncovered might give us a lead in finding out to what extent quantum dissipative processes obey the general principles of thermodynamics.

## 2. Decomposition of the dynamics into normal modes

### 2.1. The model

Resting on the electric-dipole and rotating-wave approximations, the Jaynes–Cummings Hamiltonian

$$H = \sigma_+ \otimes a + \sigma_- \otimes a^\dagger + \Delta(i_+ - i_-) \otimes \mathbf{1} \quad (1)$$

models the interaction between a two-level atom ( $A$ ) and a single privileged mode of the quantized electromagnetic radiation field ( $F$ ). The Hamiltonian acts on the product Hilbert space  $\mathbb{C}^2 \otimes \mathcal{H}_F$ , where  $\mathbb{C}^2$  is spanned by the excited state  $\hat{\mathbf{e}}_1 = (1, 0)^T$  and the ground state  $\hat{\mathbf{e}}_2 = (0, 1)^T$  of the atom.

The four matrices

$$i_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad i_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (2)$$

generate all atomic operators. The number states  $\{|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle\}_{n=0}^\infty$  span  $\mathcal{H}_F$ , the Hilbert space of the field. The ladder operators of the privileged mode are denoted by  $a$  and  $a^\dagger$ . The commutator  $[a, a^\dagger]$  is equal to unity and the state  $a|0\rangle$  is identical to the zero element of  $\mathcal{H}_F$ . Choosing the coupling constant  $g$  of the Jaynes–Cummings interaction as a scale, we use the detuning parameter  $\Delta = (\omega_A - \omega_F)/(2g)$  to measure the difference between the frequency  $\omega_A$  of the atomic transition and the frequency  $\omega_F$  of the field mode. This difference may not become arbitrarily large, otherwise the rotating-wave approximation is violated.

We introduce dissipation by connecting both the atom and the field to a thermal reservoir of Markovian nature. There is no interaction between the atomic and the photonic reservoirs. By way of the well-known weak-coupling procedure [10], we derive for the full density operator  $\rho(t)$  the following master equation:

$$\frac{d}{dt}\rho(t) = \mathcal{L}_1[\rho(t)] + \frac{\kappa}{1+\bar{n}}\mathcal{L}_2[\rho(t)] + \frac{2\gamma}{1-2d_\infty}\mathcal{L}_3[\rho(t)]. \quad (3)$$

We have moved to the interaction picture, and divided by the coupling constant  $g$ . Therefore, the time  $t$ , the strength  $\kappa$  of photonic damping and the strength  $\gamma$  of atomic damping are dimensionless.

As in I, the conservative term of (3) is given by

$$\mathcal{L}_1[\rho] = -i[H, \rho]. \quad (4)$$

We allow the inverse temperatures  $\beta_A$  and  $\beta_F$  of the atomic and the photonic reservoirs to be finite. Then the operator describing photon loss takes the form [11]

$$\begin{aligned} \mathcal{L}_2[\rho] = & 2(\mathbf{1}_2 \otimes a)\rho(\mathbf{1}_2 \otimes a^\dagger) - (\mathbf{1}_2 \otimes a^\dagger a)\rho - \rho(\mathbf{1}_2 \otimes a^\dagger a) + 2\bar{n}[(\mathbf{1}_2 \otimes a^\dagger) \\ & \times \rho(\mathbf{1}_2 \otimes a) + (\mathbf{1}_2 \otimes a)\rho(\mathbf{1}_2 \otimes a^\dagger) - (\mathbf{1}_2 \otimes a^\dagger a)\rho - \rho(\mathbf{1}_2 \otimes aa^\dagger)]. \end{aligned} \quad (5)$$

Parameter  $\bar{n}$  stands for the average number of thermal photons in the privileged mode. The operator governing atomic dissipation appears as [11]

$$\begin{aligned} \mathcal{L}_3[\rho] = & (1/2 - d_\infty)[2(\sigma_- \otimes \mathbf{1})\rho(\sigma_+ \otimes \mathbf{1}) - (i_+ \otimes \mathbf{1})\rho - \rho(i_+ \otimes \mathbf{1})] \\ & + (1/2 + d_\infty)[2(\sigma_+ \otimes \mathbf{1})\rho(\sigma_- \otimes \mathbf{1}) - (i_- \otimes \mathbf{1})\rho - \rho(i_- \otimes \mathbf{1})] \\ & - \Gamma[(i_+ \otimes \mathbf{1})\rho(i_- \otimes \mathbf{1}) + (i_- \otimes \mathbf{1})\rho(i_+ \otimes \mathbf{1})]. \end{aligned} \quad (6)$$

The contribution with  $\Gamma$  takes into account atomic dephasing. It is caused by collisions with residual gas molecules. Parameter  $d_\infty$  indicates the degree of atomic inversion for  $t \rightarrow \infty$ . Derivation of the Bloch equations on the basis of (3) and (6) gives the relaxation times  $T_1^{-1} = 4\gamma(1 - 2d_\infty)^{-1}$  and  $T_2^{-1} = 2\gamma(1 + \Gamma)(1 - 2d_\infty)^{-1}$ .

The dependences of  $\bar{n}$  and  $d_\infty$  on temperature follow from the relations

$$\frac{\bar{n}}{1+\bar{n}} = \exp(-\beta_F\omega_F) \equiv \lambda \quad \frac{1+2d_\infty}{1-2d_\infty} = \exp(-\beta_A\omega_A). \quad (7)$$

As long as  $\kappa$ ,  $\bar{n}$ ,  $\gamma$  and  $\Gamma$  remain non-negative, and the inequality  $|d_\infty| \leq 1/2$  is fulfilled, the master equation (3) preserves the trace, the self-adjointness and the positivity of the initial density operator  $\rho(t=0)$ .

From here onwards, we assume that the relation

$$\beta_A\omega_A = \beta_F\omega_F \quad (8)$$

holds true. Then the factorized thermal state

$$\rho_{\text{th}} = \begin{pmatrix} 1/2 + d_\infty & 0 \\ 0 & 1/2 - d_\infty \end{pmatrix} \otimes \exp(-\beta_F\omega_F a^\dagger a) / \text{Tr}_F[\exp(-\beta_F\omega_F a^\dagger a)] \quad (9)$$

commutes with the Jaynes–Cummings Hamiltonian  $H$ . Hence, the condition of detailed balance is fulfilled. This property implies that for long times the composite system of atom and field mode respects all principles of non-equilibrium thermodynamics [12]. Indeed, the thermal state (9) turns out to be a fixed point of the dynamics. We stress that for reservoirs of equal temperature the constraint (8) forces us to take the detuning parameter as zero.

### 2.2. Towards an algebraic equation

To get rid of the operator character of (3), we resort to the same strategy as in I. Decomposing the density operator as

$$\rho(t) = i_+ \otimes \rho_1(t) + \sigma_- \otimes \rho_2(t) + \sigma_+ \otimes \rho_3(t) + i_- \otimes \rho_4(t) \tag{10}$$

we gather the matrix elements of the field operators  $\{\rho_j(t)\}$  in the following manner:

$$\mathbf{v}''(t; m, n) = [\rho_1(t)_{m,n}, \rho_2(t)_{m+1,n}, \rho_3(t)_{m,n+1}, \rho_4(t)_{m+1,n+1}^T] \tag{11}$$

with  $m, n \geq -1$ . By definition, a matrix element  $\rho_j(t)_{m,n} = \langle m | \rho_j(t) | n \rangle$  vanishes as the label of the number state  $\langle m |$  or  $| n \rangle$  becomes negative.

From (3) one derives a differential equation for the new four-dimensional vector. It reads

$$\begin{aligned} \frac{d}{dt} \mathbf{v}''(t; m, n) &= A''(m, n) \mathbf{v}''(t; m, n) + S''(m, n) \mathbf{v}''(t; m + 1, n + 1) \\ &+ \lambda T''(m, n) \mathbf{v}''(t; m - 1, n - 1) \end{aligned} \tag{12}$$

with  $m, n \geq -1$ . For the specification of the  $(4 \times 4)$  matrices  $A''$ ,  $S''$  and  $T''$  we make use of the theta symbol

$$\theta_n = \begin{cases} 0 & \text{if } n \leq -1 \\ 1 & \text{if } n \geq 0. \end{cases} \tag{13}$$

Then the elements of  $A''$  can be expressed as

$$\begin{aligned} A''(m, n)_{11} &= [-\kappa(m + n) - \kappa\lambda(m + n + 2) - 2\gamma] \theta_m \theta_n \\ A''(m, n)_{22} &= [-\kappa(m + n + 1) - \kappa\lambda(m + n + 3) - \gamma(1 + \Gamma)(1 + \lambda) + 2i\Delta] \theta_{m+1} \theta_n \\ A''(m, n)_{33} &= [-\kappa(m + n + 1) - \kappa\lambda(m + n + 3) - \gamma(1 + \Gamma)(1 + \lambda) - 2i\Delta] \theta_m \theta_{n+1} \\ A''(m, n)_{44} &= [-\kappa(m + n + 2) - \kappa\lambda(m + n + 4) - 2\gamma\lambda] \theta_{m+1} \theta_{n+1} \\ A''(m, n)_{kl} &= A''(m, n)_{lk} & A''(m, n)_{14} &= A''(m, n)_{23} = 0 \\ A''(m, n)_{12} &= -i(m + 1)^{1/2} \theta_m \theta_n & A''(m, n)_{34} &= -i(m + 1)^{1/2} \theta_m \theta_{n+1} \\ A''(m, n)_{13} &= i(n + 1)^{1/2} \theta_m \theta_n & A''(m, n)_{24} &= i(n + 1)^{1/2} \theta_{m+1} \theta_n. \end{aligned} \tag{14}$$

For the elements of  $S''$  one finds

$$\begin{aligned} S''(m, n)_{11} &= 2\kappa(m + 1)^{1/2}(n + 1)^{1/2} \theta_m \theta_n \\ S''(m, n)_{22} &= 2\kappa(m + 2)^{1/2}(n + 1)^{1/2} \theta_{m+1} \theta_n \\ S''(m, n)_{33} &= 2\kappa(m + 1)^{1/2}(n + 2)^{1/2} \theta_m \theta_{n+1} \\ S''(m, n)_{44} &= 2\kappa(m + 2)^{1/2}(n + 2)^{1/2} \theta_{m+1} \theta_{n+1} \\ S''(m, n)_{41} &= 2\gamma \theta_{m+1} \theta_{n+1}. \end{aligned} \tag{15}$$

All other elements are vanishing. The matrix  $T''$  is determined by

$$T''(m, n)_{kl} = S''(m - 1, n - 1)_{lk}. \tag{16}$$

Note that the temperature enters our equations solely over the parameter  $\lambda$ . One needs the identities (7) and (8) to achieve this.

We assume that  $\rho(t)$  converges to the fixed point  $\rho_{th}$  for large times. Consequently, the limit

$$\lim_{t \rightarrow \infty} \mathbf{v}''(t; m, n) = \delta_{m,n} (1 - \lambda)(1 + \lambda)^{-1} \lambda^{n+1} (\theta_n, 0, 0, \theta_{n+1})^T \tag{17}$$

holds true. One should safeguard the validity of (17) from the very outset, i.e., already at the level of the primal equation (12). To that end, we map the right-hand side of (17) to zero for  $n \geq 0$ . This can happen by means of the transformation

$$\begin{aligned} \mathbf{v}'(t; m, n) &= \mathbf{v}''(t; m, n) \\ \mathbf{v}'(t; n, n) &= (\theta_n Q_1 + \delta_{n,-1} Q_2) \mathbf{v}''(t; n, n) - \lambda \theta_n Q_2 \mathbf{v}''(t; n-1, n-1) \end{aligned} \tag{18}$$

with  $m, n \geq -1$  and  $m \neq n$ . The matrices are chosen as

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

The inverse transformation reads

$$\mathbf{v}''(t; n, n) = [\theta_n Q_1^{-1} + \delta_{n,-1} Q_2^T] \mathbf{v}'(t; n, n) + \theta_n \sum_{k=1}^{n+1} \lambda^k Q_1^{-1} Q_2 Q_1^{-1} \mathbf{v}'(t; n-k, n-k). \tag{20}$$

Upon subjecting (12) to transformation (18), we recover (12) with all double primes replaced by single primes. The matrices carrying a single prime are presented below.

In the following subsection we shall decompose the dynamics into normal modes. To facilitate the associated matrix diagonalization, we make the split

$$A'(m, n) = A'_0(m, n) + A'_1(m, n). \tag{21}$$

Rather than  $A'$ , we shall diagonalize the matrix

$$A'_0(n, n) = \theta_n Q_1 A''(n, n; \lambda = \Gamma = 0) Q_1^{-1} \quad A'_0(m, n) = A''(m, n; \lambda = \Gamma = 0) \tag{22}$$

with  $m \neq n$ . The excess term is found as

$$\begin{aligned} A'_1(n, n)_{11} &= -2\kappa\lambda(n+1)\theta_n & A'_1(n, n)_{21} &= -2\gamma\lambda\theta_n \\ A'_1(n, n)_{22} &= [-2\kappa\lambda(n+1) - 2\gamma\lambda]\theta_n \\ A'_1(n, n)_{33} &= A'_1(n, n)_{44} = [-\kappa\lambda(2n+3) - \gamma(\lambda + \Gamma + \lambda\Gamma)]\theta_n \\ A'_1(m, n)_{11} &= -\kappa\lambda(m+n+2)\theta_m\theta_n \\ A'_1(m, n)_{22} &= [-\kappa\lambda(m+n+3) - \gamma(\lambda + \Gamma + \lambda\Gamma)]\theta_{m+1}\theta_n \\ A'_1(m, n)_{33} &= [-\kappa\lambda(m+n+3) - \gamma(\lambda + \Gamma + \lambda\Gamma)]\theta_m\theta_{n+1} \\ A'_1(m, n)_{44} &= [-\kappa\lambda(m+n+4) - 2\gamma\lambda]\theta_{m+1}\theta_{n+1} \end{aligned} \tag{23}$$

with  $m \neq n$ . It contains all contributions that are proportional to  $\lambda$  or  $\Gamma$ .

For the remaining primed matrices one has

$$\begin{aligned} S'(n, n)_{11} &= [2\kappa(n+1) - 2\gamma]\theta_n & S'(n, n)_{12} &= [-2\kappa - 2\gamma]\theta_n \\ S'(n, n)_{22} &= [2\kappa(n+2) + 2\gamma]\theta_{n+1} & S'(n, n)_{21} &= 2\gamma\theta_{n+1} \\ S'(n, n)_{33} &= S'(n, n)_{44} = 2\kappa(n+1)^{1/2}(n+2)^{1/2}\theta_n \\ T'(n, n)_{11} &= T'(n, n)_{22} = 2\kappa n\theta_{n-1} & T'(n, n)_{23} &= -in^{1/2}\theta_{n-1} \\ T'(n, n)_{33} &= T'(n, n)_{44} = 2\kappa n^{1/2}(n+1)^{1/2}\theta_{n-1} \\ S'(m, n) &= S''(m, n) & T'(m, n) &= T''(m, n) \end{aligned} \tag{24}$$

with  $m \neq n$ . The matrix elements, which are not fixed by (23) and (24), are identical to zero.

The transformed differential equation (12) can be cast into algebraic form by means of the Laplace transform

$$\tilde{f}(z) = -i \int_0^\infty dt e^{izt} f(t) \quad f(t) = \frac{i}{2\pi} \int_C dz e^{-izt} \tilde{f}(z) \tag{25}$$

where  $\text{Im } z$  is positive. We obtain the following result using (21):

$$[z\mathbf{1}_4 - iA'_0(m, n)]\tilde{\mathbf{v}}'(z; m, n) = \mathbf{v}'(t = 0; m, n) + iS'(m, n)\tilde{\mathbf{v}}'(z; m + 1, n + 1) + i\lambda T'(m, n)\tilde{\mathbf{v}}'(z; m - 1, n - 1) + iA'_1(m, n)\tilde{\mathbf{v}}'(z; m, n) \tag{26}$$

with  $m, n \geq -1$ . The master equation (3) has been converted to a four-dimensional recursion relation of second order. The iterative solution can be readily constructed [3, 8]. No problems with convergence occur as long as the parameter  $\lambda$  stays below a certain bound. For experiments in a microwave cavity, temperatures of the order of 0.1 K are admissible [8].

### 2.3. Diagonalization of the frequency matrix

In the dynamics of the damped Jaynes–Cummings model two different timescales can be distinguished.

The first one relates to the time  $g^{-1}$  that is needed for energy exchange between the atom and field mode; the second one relates to the times  $(g\kappa)^{-1}$  and  $(g\gamma)^{-1}$  for which energy losses to the reservoirs become sizeable. We can put these simple qualitative ideas into mathematical shape by subjecting the recursion (26) to a transformation that diagonalizes the frequency matrix  $[z\mathbf{1}_4 - iA'_0(m; n)]$ . We then make the poles of the Laplace transform  $\tilde{\mathbf{v}}'$  visible. By Jordan’s lemma, the inverse Laplace transformation comes down to summing up the residues of all poles. One may say therefore, that each pole generates a normal mode of the dynamics.

The diagonalization procedure relies on the identity

$$A''(m, n; \lambda = \Gamma = 0) = I(n) \otimes B(m) + B(n)^\dagger \otimes I(m) \tag{27}$$

where the usual right-Kronecker direct product has been employed [13]. The matrices on the right-hand side are defined as

$$B(n) = \begin{pmatrix} (-\kappa n - \gamma)\theta_n & -i(n + 1)^{1/2}\theta_n \\ -i(n + 1)^{1/2}\theta_n & (-\kappa n - \kappa + 2i\Delta)\theta_{n+1} \end{pmatrix} \quad I(n) = \begin{pmatrix} \theta_n & 0 \\ 0 & \theta_{n+1} \end{pmatrix}. \tag{28}$$

The eigenvalues and normalized eigenvectors of  $B(n)$  have the following neat structure:

$$\begin{aligned} \mu_\eta(n) &= [-\kappa n - \kappa/2 - \gamma/2 + i\Delta - i\eta u(n)]\theta_n + 2i\Delta\delta_{n,-1}\delta_{\eta,-1} \\ \hat{\mathbf{q}}_\eta(n) &= v_\eta^{-1}(n) \begin{pmatrix} (n + 1)^{1/2} \\ \eta u(n) - i\kappa/2 + i\gamma/2 - \Delta \end{pmatrix} \theta_n + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_{n,-1}\delta_{\eta,-1} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta_{n,-1}\delta_{\eta,+1}. \end{aligned} \tag{29}$$

The abbreviations

$$\begin{aligned} v_\eta(n) &= [n + 1 + |\eta u(n) - i\kappa/2 + i\gamma/2 - \Delta|^2]^{1/2} \\ u(n) &= [n + 1 - (\kappa/2 - \gamma/2 - i\Delta)^2]^{1/2} \end{aligned} \tag{30}$$

have been used, with  $\eta = \pm 1$ . The real part of  $u(n)$  is chosen to be positive.

The diagonalization can be effectuated with the help of the transformation

$$U(m, n) = [\delta_{m,n}\theta_n Q_1 + (1 - \delta_{m,n}\theta_n)\mathbf{1}_4] \begin{pmatrix} \hat{q}_+^*(n)_1 & \hat{q}_-^*(n)_1 \\ \hat{q}_+^*(n)_2 & \hat{q}_-^*(n)_2 \end{pmatrix} \otimes \begin{pmatrix} \hat{q}_+(m)_1 & \hat{q}_-(m)_1 \\ \hat{q}_+(m)_2 & \hat{q}_-(m)_2 \end{pmatrix}. \tag{31}$$

The matrix  $U$  is not unitary, but computation of its inverse poses no problems of course. If vectors and matrices are transformed according to

$$\begin{aligned} \mathbf{v}(t; m, n) &= U^{-1}(m, n)\mathbf{v}'(t; m, n) \\ A_0(m, n) &= U^{-1}(m, n)A'_0(m, n)U(m, n) \\ A_1(m, n) &= U^{-1}(m, n)A'_1(m, n)U(m, n) \\ S(m, n) &= U^{-1}(m, n)S'(m, n)U(m + 1, n + 1) \\ T(m, n) &= U^{-1}(m, n)T'(m, n)U(m - 1, n - 1) \end{aligned} \tag{32}$$



then the validity of (26) is not affected by a suppression of all primes. The transformed frequency matrix is indeed diagonal. Its entries follow from

$$\begin{aligned}
 A_0(m, n)_{11} &= [\mu_+(m) + \mu_+^*(n)]\theta_m\theta_n \\
 A_0(m, n)_{22} &= [\mu_-(m) + \mu_+^*(n)]\theta_{m+1}\theta_n \\
 A_0(m, n)_{33} &= [\mu_+(m) + \mu_-^*(n)]\theta_m\theta_{n+1} \\
 A_0(m, n)_{44} &= [\mu_-(m) + \mu_-^*(n)]\theta_{m+1}\theta_{n+1}.
 \end{aligned} \tag{33}$$

The other matrices figuring in (32) will be evaluated in so far as required.

From the last result the location of the poles of  $\tilde{v}(z; m, n)$  can be read off. All normal modes suffer from dissipation, the case  $m = n = -1$  excepted. The matrix  $A_0(-1, -1)$  is vanishing, so for  $j = 2$  the vector component  $\tilde{v}(z; -1, -1)_j$  has a simple pole at  $z = 0$ . By definition, for  $j \neq 2$  this component equals zero. In carrying out Laplace backtransformation, the pole at  $z = 0$  gives rise to a residue that, in contrast to all other residues, does not depend on time. Hence, verification of limit (17) takes place by simply discarding all damped modes. Now the advantage of the temperature transformation becomes manifest. If we refrain from carrying out (18), then all poles become dissipative. It is no longer clear which ones may be dropped for large times. In [8] a rigorous check of (17) has been made for the case of pure photonic damping, i.e.,  $\gamma$  equal to zero.

Having completed the diagonalization of the frequency matrix, we can commence with an analysis of the normal modes. Moreover, we can try to select those modes that are of interest if it comes to establishing a limit of maximum entropy for the two-level atom. This is the subject of the following section.

### 3. Separation of timescales by means of projection

#### 3.1. Definition of projectors

If we linearize in  $\kappa$  and  $\gamma$ , the eigenvalue (29) takes on the form

$$\mu_\eta(n) = -i\eta(n+1 + \Delta^2)^{1/2} + i\Delta - \kappa n - (\kappa + \gamma)/2 + \eta\Delta(n)(\kappa - \gamma)/2 \tag{34}$$

where  $n$  is non-negative and the notation

$$\Delta(n) = \Delta(n+1 + \Delta^2)^{-1/2} \tag{35}$$

appears. Insertion of (34) into (33) yields

$$\begin{aligned}
 A_0(n, n)_{11} &= -2\kappa n - \kappa - \gamma + \Delta(n)(\kappa - \gamma) \\
 A_0(n, n)_{22} &= 2i(n+1 + \Delta^2)^{1/2} - 2\kappa n - \kappa - \gamma \\
 A_0(n, n)_{33} &= -2i(n+1 + \Delta^2)^{1/2} - 2\kappa n - \kappa - \gamma \\
 A_0(n, n)_{44} &= -2\kappa n - \kappa - \gamma - \Delta(n)(\kappa - \gamma)
 \end{aligned} \tag{36}$$

for  $n \geq 0$ . From these expressions it is obvious that the density operator of the damped Jaynes–Cummings model evolves at two different paces.

The first and fourth diagonal elements describe modes of a purely dissipative character, which undergo slow variations on the timescales of  $(g\kappa)^{-1}$  and  $(g\gamma)^{-1}$ . The two other diagonals correspond to oscillating modes. Being driven by the interaction time  $g^{-1}$  of the Jaynes–Cummings Hamiltonian, these modes are of fast variation. We shall see that they are of minor importance for the derivation of a limit of maximum entropy.

The separation of timescales, as displayed by the pole structure of the solution vector  $\tilde{v}(z; n, n)$ , can be transferred to the unprimed recursion (26) without much effort. We let

ourselves to be guided by well-known projection techniques, which were developed for investigating the collective modes of fluids [9]. The findings of (36) invite us to define the projectors

$$P_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad P_f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{37}$$

and decompose the solution vector as

$$\mathbf{v}(t; n, n) = \mathbf{v}_s(t; n, n) + \mathbf{v}_f(t; n, n) \quad \mathbf{v}_j(t; n, n) = P_j \mathbf{v}(t; n, n) \tag{38}$$

with  $n \geq 0$  and  $j = s, f$ .

We claim that  $\mathbf{v}_s(t; n, n)$  is a slow component, and that  $\mathbf{v}_f(t; n, n)$  is a fast component. This is confirmed by writing down the equations of motion. With the help of the identities

$$P_s A_0(n, n) P_f = P_f A_0(n, n) P_s = 0 \tag{39}$$

we deduce from the unprimed version of (26)

$$\begin{aligned} \tilde{\mathbf{v}}_s(z; n, n) = & P_s [z P_s - i P_s A_0(n, n) P_s]^{-1} P_s [\mathbf{v}_s(t = 0; n, n) \\ & + i P_s S(n, n) P_s \tilde{\mathbf{v}}_s(z; n + 1, n + 1) + i \lambda P_s T(n, n) P_s \tilde{\mathbf{v}}_s(z; n - 1, n - 1) \\ & + i P_s A_1(n, n) P_s \tilde{\mathbf{v}}_s(z; n, n) + i P_s S(n, n) P_f \tilde{\mathbf{v}}_f(z; n + 1, n + 1) \\ & + i \lambda P_s T(n, n) P_f \tilde{\mathbf{v}}_f(z; n - 1, n - 1) + i P_s A_1(n, n) P_f \tilde{\mathbf{v}}_f(z; n, n)] \end{aligned} \tag{40}$$

with  $n$  non-negative. To obtain a closed set, the interchange  $s \leftrightarrow f$  must be performed. This provides us with the fast counterpart of (40). Note that the poles of the resolvent  $P_{s(f)} [z P_{s(f)} - i P_{s(f)} A_0(n, n) P_{s(f)}]^{-1} P_{s(f)}$  generate residues of slow (fast) variation, so  $\mathbf{v}_{s(f)}(t; n, n)$  is indeed a slow (fast) component.

For the special case of  $n = -1$ , equation (26) reduces to

$$\tilde{\mathbf{v}}(z; -1, -1) = z^{-1} [\mathbf{v}(t = 0; -1, -1) + i S(-1, -1) \tilde{\mathbf{v}}(z; 0, 0)]. \tag{41}$$

One should remember that  $T(-1, -1)$  and  $A_{0,1}(-1, -1)$  equal zero. To process (26) for the case of  $m \neq n$ , we return to expansion (34). It tells us that now all diagonals of  $A_0$  remain finite as  $\kappa$  and  $\gamma$  become small. Consequently, a slow component of  $\mathbf{v}$  does not exist. For treatment of the case  $m \neq n$  the result

$$\begin{aligned} \tilde{\mathbf{v}}(z; m, n) = & I(n) \otimes I(m) [z \mathbf{1}_4 - i A_0(m, n)]^{-1} I(n) \otimes I(m) [\mathbf{v}(t = 0; m, n) \\ & + i S(m, n) \tilde{\mathbf{v}}(z; m + 1, n + 1) + i \lambda T(m, n) \tilde{\mathbf{v}}(z; m - 1, n - 1) \\ & + i A_1(m, n) \tilde{\mathbf{v}}(z; m, n)] \end{aligned} \tag{42}$$

is available. One should recognize that the vanishing of  $\rho_j(t)_{m,n}$  for negative photon number  $m$  or  $n$  leads to the identity

$$I(n) \otimes I(m) \mathbf{v}(t; m, n) = \mathbf{v}(t; m, n) \tag{43}$$

with  $m \neq n$ . It has been employed in (42).

### 3.2. Limit of weak damping

We are ready to carry out the separation of timescales in (40). It is our aim to decouple the recursion for  $\tilde{\mathbf{v}}_s(z; n, n)$  from that for  $\tilde{\mathbf{v}}_f(z; n, n)$ . To that end, we make  $\kappa$  and  $\gamma$  small by taking the following limit of weak damping:

$$\kappa = \kappa' / \xi \quad \gamma = \gamma' / \xi. \tag{44}$$

The auxiliary parameter  $\xi$  tends to infinity, whereas  $\kappa'$  and  $\gamma'$  remain finite.

We remark that from here onwards, our treatment is no longer mathematically rigorous. Contributions of first order in  $1/\xi$  will be neglected against contributions of zeroth order. In principle, simplifications of this kind must be supported by mathematical estimates. These should demonstrate that all errors converge to zero as  $\xi$  becomes large. For the case of pure photonic losses, it is indeed possible to put the limit of weak damping on a firm mathematical basis [3, 8]. One could try to repeat that for the present setting. The disadvantage is that a construction of the complete solutions of (40) cannot be avoided. We cut this time-consuming path short by sorting out the implications of weak damping directly in (40).

According to (36), the poles of  $\tilde{\mathbf{v}}_s(z; n, n)$  are of order  $1/\xi$ . Hence, the vector  $\mathbf{v}_s(t; n, n)$  depends on time through the ratio  $t/\xi$ , something which can be stated explicitly via introduction of the scaling  $z = y/\xi$  into the Laplace backtransform (25). One then concludes that the slow vector

$$\mathbf{w}_s(y; n) = \xi^{-1} \tilde{\mathbf{v}}_s(y/\xi; n, n) \tag{45}$$

is of order unity for  $n \geq 0$  and  $\xi$  large. The same is true for the fast vector

$$\mathbf{w}_f(z; n) = \tilde{\mathbf{v}}_f(z; n, n) \tag{46}$$

because the poles of  $\tilde{\mathbf{v}}_f(z; n, n)$  are of order unity for  $n \geq 0$  and  $\xi$  large.

The equations of motion for the vectors (45) and (46) are found upon taking  $\xi$  to infinity in (40). It is assumed that: (i) one may drop contributions of order  $1/\xi$ , such as  $\tilde{\mathbf{v}}_{s,f}(z; n, n)/\xi$  and  $\tilde{\mathbf{v}}_f(y/\xi; n, n)/\xi$ , with  $z$  and  $y$  fixed; (ii) one may take the limits

$$M(n)_{(js)} = \lim_{\xi \rightarrow \infty} \xi P_j M(n, n) P_s \quad M(n)_{(jf)} = \lim_{\xi \rightarrow \infty} P_j M(n, n) P_f \tag{47}$$

with  $j = s, f$  and  $M = A_0, A_1, S, T$ . Since the matrices  $A_1$  and  $S$  are of order  $1/\xi$ , they do not survive the second limit. One furthermore checks that the limiting matrix  $T(n)_{(fs)}$  is identical to zero.

With the above guidelines in mind, one derives from the fast counterpart of (40) the simple relation

$$\mathbf{w}_f(z; n) = P_f [z P_f + i(2\kappa n + \kappa + \gamma) P_f - iA_0(n)_{(ff)}]^{-1} P_f \mathbf{v}_f(t = 0; n, n) \tag{48}$$

where  $n$  is non-negative. In zeroth order of  $1/\xi$  we do not meet terms containing the slow component. In the resolvent the contributions with  $\kappa$  and  $\gamma$  should be retained, otherwise the asymptotic limit (17) is compromised.

Equation (40) itself must be handled carefully. Due to the fact that  $T(n, n)$  is of order unity, we meet the fast vector  $\tilde{\mathbf{v}}_f(y/\xi; n - 1, n - 1)$ . With the aid of the fast counterpart of (40), we can exchange it for an expression that solely depends on (45). In doing so, we arrive for  $n \geq 0$  at

$$\begin{aligned} \mathbf{w}_s(y; n) = & P_s [y P_s - iA_0(n)_{(ss)}]^{-1} P_s [\mathbf{v}_s(t = 0; n, n) \\ & + iS(n)_{(ss)} \mathbf{w}_s(y; n + 1) + i\lambda T(n)_{(ss)} \mathbf{w}_s(y; n - 1) \\ & + iA_1(n)_{(ss)} \mathbf{w}_s(y; n) - \lambda \theta_{n-1} T(n)_{(sf)} A_0^{-1}(n - 1)_{(ff)} \mathbf{v}_f(t = 0; n - 1, n - 1) \\ & - i\lambda \theta_{n-1} T(n)_{(sf)} A_0^{-1}(n - 1)_{(ff)} S(n - 1)_{(fs)} \mathbf{w}_s(y; n) \\ & - i\lambda \theta_{n-1} T(n)_{(sf)} A_0^{-1}(n - 1)_{(ff)} A_1(n - 1)_{(fs)} \mathbf{w}_s(y; n - 1)]. \end{aligned} \tag{49}$$

The identity  $T(n)_{(fs)} = 0$  ensures that the right-hand side does not include a term containing  $\mathbf{w}_s(y; n - 2)$ .

Next, we investigate the behaviour of (41) and (42) under the limit of weak damping. By performing inverse Laplace transformation, and taking advantage of the fact that  $S(-1, -1)$  is of order  $1/\xi$ , we can reshape (41) as follows:

$$\mathbf{v}(t; -1, -1) = \mathbf{v}(t = 0; -1, -1) + \oint \frac{dy}{2\pi} y^{-1} \exp(-iyt/\xi) S(-1)_{(fs)} \mathbf{w}_s(y; 0) \tag{50}$$

where  $\xi$  must be chosen large. Because of the limit  $S(-1)_{(ff)} = 0$ , the vector  $\tilde{\mathbf{v}}_f(z; 0, 0)$  does not contribute.

We recall that  $A_0(m, n)$  is of zeroth order for  $m \neq n$ . On the other hand,  $A_1(m, n)$ ,  $S(m, n)$  and  $T(m, n)$  are of order  $1/\xi$ . We therefore may replace (42) by

$$\tilde{\mathbf{v}}(z; m, n) = I(n) \otimes I(m) [z\mathbf{1}_4 + i(\kappa m\theta_m + \kappa n\theta_n + \kappa/2 + \gamma/2)\mathbf{1}_4 - iA_0(m, n; \kappa = \gamma = 0)]^{-1} I(n) \otimes I(m) \mathbf{v}(t = 0; m, n) \tag{51}$$

with  $m \neq n$  and  $\xi \gg 1$ . The terms proportional to  $\kappa$  and  $\gamma$  ensure that Laplace backtransformation yields a result which decays to zero for large times.

The last remark implies that the limit of weak damping leaves the asymptotic behaviour of the density operator intact. We make an explicit check for the case of  $\lambda = \Delta = \Gamma = 0$ . All poles generate damped residues, except for the pole at  $y = 0$  in (50). This observation brings us to the relation

$$\lim_{t \rightarrow \infty} \mathbf{v}(t; m, n) = \delta_{m,-1} \delta_{n,-1} [\mathbf{v}(t = 0; -1, -1) + iS(-1)_{(fs)} \mathbf{w}_s(0; 0)]. \tag{52}$$

We can eliminate  $\mathbf{w}_s(0; 0)$  by writing down the iterative solution of (49). This gives rise to the relation

$$\mathbf{w}_s(0; 0) = i \sum_{k=0}^{\infty} \left[ \prod_{l=0}^{k-1} -A_0^{-1}(l)_{(ss)} S(l)_{(ss)} \right] A_0^{-1}(k)_{(ss)} \mathbf{v}_s(t = 0; k, k). \tag{53}$$

The right-hand side can be computed on the basis of the prescriptions (32) and (47). In elaborating the matrix product, one must choose the order  $0, 1, 2, \dots, k - 1$  for the index  $l$ . The right-hand side of (52) reduces to  $\delta_{m,-1} \delta_{n,-1} (0, 1, 0, 0)^T$ , a result that confirms (17) for  $\lambda = \Delta = \Gamma = 0$ .

Equations (48)–(51) fully determine the density operator  $\rho(t)$  in the limit of weak damping. The corresponding dynamics consists of two independent components, a slow one and a fast one. The latter exhibits oscillations on the timescale of  $g^{-1}$ . This feature rules out the existence of any stable fixed points. Hence, in our search for a limit of maximum entropy we must set our hopes on the normal modes of slow variation. But before commencing this programme, we should make an important remark on the choice of the photon number  $n$ . If one employs the limits (47), then one tacitly assumes that the ratio  $n/\xi$  tends to zero for  $\xi$  large. Therefore, an increase of  $n$  is permitted only if  $n/\xi$  is kept small.

#### 4. Evolution of the atomic density matrix

##### 4.1. Calculation of matrix elements

As mentioned in section 1, the goal of this paper is to study the atomic density matrix

$$\rho_A(t) = \text{Tr}_F[\rho(t)]. \tag{54}$$

More specifically, we are going to devise a limit under which  $\rho_A(t)$  converges to the central state  $\mathbf{1}_2/2$ . Then the von Neumann entropy of the atom, given by  $-\text{Tr}_A[\rho_A \log \rho_A]$ , attains its maximum value of  $\log 2$ .

By (10), (11) and (20) the entries of the atomic density matrix can be calculated from

$$\begin{aligned} \rho_A(t)_{22} &= 1 - \rho_A(t)_{11} = (1 + \bar{n}) \sum_{n=-1}^{\infty} \mathbf{v}'(t; n, n)_2 \\ \rho_A(t)_{12} &= \rho_A(t)_{21}^* = \sum_{n=0}^{\infty} \mathbf{v}'(t; n, n - 1)_3. \end{aligned} \tag{55}$$

We have made use of the fact that  $\rho_A(t)$  is Hermitian, and that its trace is equal to unity. The primes can be removed through application of transformation (32). From I, we already know that  $\xi$  becomes large in the limit of maximum entropy. Hence, in working out (55) we may appeal to the findings of the previous section.

Upon substituting the results (48) and (51) into the Laplace backtransform (25), we learn that

$$\begin{aligned}
 (1 + \bar{n})^{-1} \rho_A(t)_{22} = & \mathbf{v}(t = 0; -1, -1)_2 + \left[ \oint \frac{dy}{2\pi} y^{-1} \exp(-iyt/\xi) S(-1)_{(fs)} \mathbf{w}_s(y; 0) \right]_2 \\
 & + \left[ \sum_{n=0}^{\infty} \exp(-2\kappa tn - \kappa t - \gamma t) U(n, n) \oint \frac{dz}{2\pi i} \exp(-izt) P_f \right. \\
 & \times [zP_f - iA_0(n)_{(ff)}]^{-1} P_f \mathbf{v}_f(t = 0; n, n) \left. \right]_2 \\
 & + \left[ \sum_{n=0}^{\infty} U(n, n) \oint \frac{dy}{2\pi i} \exp(-iyt/\xi) \mathbf{w}_s(y; n) \right]_2 \tag{56}
 \end{aligned}$$

$$\begin{aligned}
 \rho_A(t)_{12} = & \left[ \sum_{n=0}^{\infty} U(n, n-1) \exp[-\kappa tn - \kappa t(n-1)\theta_{n-1} - (\kappa + \gamma)t/2] \oint \frac{dz}{2\pi i} \exp(-izt) \right. \\
 & \times I(n-1) \otimes I(n) [z\mathbf{1}_4 - iA_0(n, n-1; \kappa = \gamma = 0)]^{-1} \\
 & \times I(n-1) \otimes I(n) \mathbf{v}(t = 0; n, n-1) \left. \right]_3 \tag{57}
 \end{aligned}$$

As pointed out, these results are valid for  $\xi \gg 1$ , so one may also expand the transformation  $U(m, n)$ . In the second contribution of (56), as well as in (57), the integration variable  $z$  has been transformed such that damping factors explicitly appear.

The last contribution of (56) originates from the modes of slow variation. It can be evaluated in a convenient manner by proposing the expansion

$$\mathbf{w}_s(y; n) = \sum_{l=1}^{\infty} y^{-l} P_s \tilde{\mathbf{b}}(l, n) \tag{58}$$

with  $n \geq 0$ . This is equivalent to expanding the exponential factor of the Laplace backtransform. Substitution of (58) into (49) produces the following recursion in  $l$  and  $n$ :

$$\begin{aligned}
 P_s \tilde{\mathbf{b}}(l, n) = & [iA_0(n)_{(ss)}]^{l-1} P_s \tilde{\mathbf{b}}(1, n) + i \sum_{q=1}^{l-1} [iA_0(n)_{(ss)}]^{q-1} S(n)_{(ss)} P_s \tilde{\mathbf{b}}(l-q, n+1) \\
 & + i\lambda \sum_{q=1}^{l-1} [iA_0(n)_{(ss)}]^{q-1} T(n)_{(ss)} P_s \tilde{\mathbf{b}}(l-q, n-1) \\
 & + i \sum_{q=1}^{l-1} [iA_0(n)_{(ss)}]^{q-1} A_1(n)_{(ss)} P_s \tilde{\mathbf{b}}(l-q, n) \\
 & - i\lambda \theta_{n-1} \sum_{q=1}^{l-1} [iA_0(n)_{(ss)}]^{q-1} T(n)_{(sf)} A_0^{-1}(n-1)_{(ff)} S(n-1)_{(fs)} P_s \tilde{\mathbf{b}}(l-q, n) \\
 & - i\lambda \theta_{n-1} \sum_{q=1}^{l-1} [iA_0(n)_{(ss)}]^{q-1} T(n)_{(sf)} A_0^{-1}(n-1)_{(ff)} \\
 & \times A_1(n-1)_{(fs)} P_s \tilde{\mathbf{b}}(l-q, n-1) \tag{59}
 \end{aligned}$$

where the conditions  $l \geq 1$  and  $n \geq 0$  must be imposed. For the vector with  $l = 1$  the result  $P_s \tilde{\mathbf{b}}(1, n) = \mathbf{v}_s(t = 0; n, n) - \lambda \theta_{n-1} T(n)_{(sf)} A_0^{-1} (n-1)_{(ff)} \mathbf{v}_f(t = 0; n-1, n-1)$  (60) is found.

A simple iteration in integer  $l$  furnishes the solution of (59). At the same time, the ensuing expressions are rather lengthy. If the detuning parameter is set equal to zero, the going becomes much easier. From equations (29)–(31) one derives

$$U(n, n) = U_0 + \frac{i(\gamma' - \kappa')}{2\xi(n+1)^{1/2}} U_1$$

$$U_0 = \frac{1}{2} \begin{pmatrix} 0 & 2 & 2 & 0 \\ 1 & -1 & -1 & 1 \\ 0 & -2 & 2 & 0 \\ 2 & 0 & 0 & -2 \end{pmatrix} \quad U_1 = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (61)$$

up to first order in  $1/\xi$ , and for  $n$  non-negative.

All matrices figuring in (59) can be calculated now with relative ease. One should employ (23), (24), (32) and (36). Note that  $U_1$  makes a contribution to  $T(n)_{(ss)}$ . In view of (56) it is meaningful to switch to the new vectors

$$\mathbf{b}(l, n) = i^{l-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U_0 P_s \tilde{\mathbf{b}}(l, n) \quad (62)$$

with  $l \geq 1$  and  $n \geq 0$ . Still keeping  $\Delta$  zero, one succeeds in extracting from (59) the less involved recursion

$$\mathbf{b}(l, n) = (2\kappa' n + \kappa' + \gamma')^{l-1} \mathbf{b}(1, n) - \sum_{q=1}^{l-1} (2\kappa' n + \kappa' + \gamma')^{q-1} \times \{[\kappa'(2n+3) + \gamma'] \mathbf{b}(l-q, n+1) + \lambda \theta_{n-1} [\kappa'(2n-1) + \gamma'] \mathbf{b}(l-q, n-1) - \lambda [\kappa'(2n+2 - \theta_{n-1}) + \gamma'(2 - \theta_{n-1})] \mathbf{b}(l-q, n)\}. \quad (63)$$

Use of (62) in (60) gives

$$\mathbf{b}(1, n)_j = \frac{1}{2} \delta_{j,2} [\mathbf{v}(t = 0; n, n)_1 + \mathbf{v}(t = 0; n, n)_4 - \lambda \theta_{n-1} \mathbf{v}(t = 0; n-1, n-1)_2 - \lambda \theta_{n-1} \mathbf{v}(t = 0; n-1, n-1)_3] \quad (64)$$

with  $j = 1, 2, 3, 4$ . Note that the parameter  $\Gamma$  for atomic dephasing has disappeared from our equations.

In solving (63), the above equality serves as the initial condition. One may write

$$\mathbf{b}(l, n) = \sum_{s=n-l+1}^{n+l-1} \theta_s d(l, n, s) \mathbf{b}(l = 1, s) \quad (65)$$

with  $l \geq 1$  and  $n \geq 0$ . As is evident from (56), the sum

$$\sum_{n=0}^{\infty} \mathbf{b}(l, n) = \sum_{s=0}^{\infty} \mathbf{b}(1, s) \sum_{n=s-l+1}^{s+l-1} \theta_n d(l, n, s) \quad (66)$$

should be computed. We can prove the following remarkable identity:

$$\sum_{n=s-l}^{s+l} \theta_n d(l+1, n, s) = (1 + \lambda)(\kappa' + \gamma') d(l, n = 0, s) \theta_{l-s-1}. \quad (67)$$

The integers  $l - 1$  and  $s$  are non-negative. In principle, the coefficient  $d(l, n = 0, s)$  remains to be calculated. Fortunately, we can do without this degree of precision.

The proof of (67) relies on the recursion relation for the coefficients  $\{d(l, n, s)\}$ , which can be derived by substituting (65) into (63). In the source term a Kronecker delta  $\delta_{n,s}$  is contained. On the left-hand side of (67) one makes use of the recursion for  $d(l + 1, n, s)$ . In eliminating the source term with  $\delta_{n,s}$ , one resorts to the recursion for  $d(l, n, s)$ . Upon performing suitable shifts of summation indices, one is left with the right-hand side of (67).

4.2. Convergence to the central state

Having completed all necessary computations, we can pass over to choosing the initial density operator  $\rho(t = 0; \alpha)$ . We assume a dependence on the positive parameter  $\alpha$ . For a coherent initial state,  $\alpha$  would be the norm of the coherence parameter. However, the details of  $\rho(t = 0; \alpha)$  need not be specified. This operator should merely obey the constraint

$$\lim_{\alpha \rightarrow \infty} \langle m | \rho_j(t = 0; \alpha) | n \rangle = 0 \tag{68}$$

for  $j = 1, 2, 3, 4$ , where we refer to the decomposition (10). The labels of the number states  $|m\rangle$  and  $|n\rangle$  must be kept fixed. Note that one may choose initial states for which the atom and field are entangled.

Although the above assumption is not strong at all, it has a striking effect on formulae (56) and (57). From the definition of  $\mathbf{v}(t; m, n)$  we infer that for  $m$  and  $n$  fixed the limit

$$\lim_{\alpha \rightarrow \infty} \mathbf{v}(t = 0; m, n) = 0 \tag{69}$$

holds true. The recursions (59) and (60) tell us that for  $l$  and  $n$  fixed we may also exploit the limit

$$\lim_{\alpha \rightarrow \infty} \tilde{\mathbf{b}}(l, n) = 0. \tag{70}$$

The tilde may be omitted if desired.

In (56) and (57) we let both  $\alpha$  and  $\xi$  tend to infinity. We wish the product  $\kappa t$  to stay finite, so (44) forces us to set  $t = t' \xi$ , with  $\kappa' > 0$  and  $t' > 0$ . By (69), the summand of (57) converges to zero for  $\alpha \rightarrow \infty$ , at least, as long as  $n \leq n_0$ , with  $n_0$  fixed. In front of the remaining summation an exponential factor of  $\exp(-\kappa' t' n_0)$  figures. The latter can become arbitrarily small, because there is no bound on the integer  $n_0$ . As a consequence, the off-diagonal  $\rho_A(t)_{12}$  decays to zero for  $\alpha$  and  $\xi$  large. A similar argument can be applied to the summation in (56) that depends on the fast modes.

The handling of the slow terms of (56) requires substitution of the expansion (58). We shall interchange the sum over  $l$  with the limit of  $\alpha, \xi \rightarrow \infty$ . This step can be justified for the special case of  $\gamma = 0, \alpha = q$  and  $\rho(t = 0; \alpha) = \rho_A \otimes |q\rangle\langle q|$ , where  $|q\rangle$  denotes a number state [8].

The term containing  $\mathbf{w}_s(y; 0)$  essentially reduces to the sum  $\sum_{l=1}^{\infty} (-it')^l \tilde{\mathbf{b}}(l, 0) / l!$ , which converges to zero for  $\alpha \rightarrow \infty$ . Use should be made of the property (70) and the interchange that has been given above. All in all, formulae (56) and (57) clear up a lot as soon as one invokes the assumption (68).

We still have to process the third summation on the right-hand side of (56). Now the identities (66) and (67) come into play. We are led to

$$\rho_A(t)_{22} = (1 + \bar{n}) \sum_{n=0}^{\infty} \mathbf{b}(1, n)_2 + (1 + 2\bar{n})(\kappa' + \gamma') \sum_{l=1}^{\infty} (-t')^l \sum_{s=0}^{l-1} \mathbf{b}(1, s)_2 d(l, 0, s) / l! \tag{71}$$

where  $\alpha$  and  $\xi$  must be chosen large. Once more we let  $\alpha$  tend to infinity behind a sum over  $l$ . Then, on account of (70), the second contribution vanishes. Evaluation of the first contribution is straightforward. Employing the fact that the trace of  $\rho(0; \alpha)$  is normalized to one, we find that  $\rho_A(t)_{22}$  converges to  $1/2$  for  $\alpha, \xi \rightarrow \infty$ .

Before presenting our limit of maximum entropy, we are obliged to discuss the relation between the large parameters  $\alpha$  and  $\xi$ . This relation stems from the fact that terms of order  $n/\xi$  were neglected during the derivation of equations (48)–(51). Before closing the previous section, we already pointed out that increase in the photon number  $n$  should be carefully observed. The computation of the atomic density matrix allows for such an increase, because a sum over all photon numbers is performed. It is far from simple to come up with a sufficient condition that supports the disregard of contributions of order  $n/\xi$ . We shall have to content ourselves with some qualitative considerations.

If we refrain from taking the limit of weak damping, then we must compute the density matrix  $\rho_A(t)$  on the basis of (40) and (42). Now the summands of (55) include contributions of order  $n/\xi$ . On the other hand, the initial vectors  $\{\mathbf{v}''(t = 0; n, n)\}$  still act as weights. Apparently, the limit of weak damping stands or falls with the assertion that terms of order

$$\sum_{n=1}^{\infty} \mathbf{v}''(t = 0; n, n)_j n/\xi \sim \epsilon_F/\xi \tag{72}$$

may be dropped. The dimensionless energy density of the initial field is defined as

$$\epsilon_F = \text{Tr}[a^\dagger a \rho(t = 0; \alpha)]. \tag{73}$$

By elaborating the trace with the help of number states, one roughly recovers the left-hand side of (72) for  $j = 1$  and  $j = 4$ . The two other values of  $j$  do not matter. The positivity of  $\rho(t = 0; \alpha)$  implies that

$$|\mathbf{v}''(t = 0; n, n)_j|^2 \leq \mathbf{v}''(t = 0; n, n)_1 \mathbf{v}''(t = 0; n, n)_4 \tag{74}$$

for  $j = 2, 3$ .

It is important to recognize that  $\epsilon_F$  becomes infinitely large under the limit of  $\alpha \rightarrow \infty$ . This follows from (68) and the normalization  $\text{Tr}[\rho(t = 0; \alpha)] = 1$ . Hence, the assertion (72) prescribes that the limit of maximum entropy must be taken under the constraint  $\xi = (\epsilon_F)^\delta$ , with  $\delta > 1$ . To fix  $\delta$  we appeal to an earlier work for the case of pure photonic damping, that is to say, the case  $\gamma = 0$ . Taking a number state as the initial condition for the density operator, one demonstrates that the choice  $\delta = 3$  constitutes a sufficient condition for discarding the order of  $n/\xi$  [8]. We assume that  $\delta$  need not be augmented if the atomic reservoir is activated again. Therefore, the relation between  $\alpha$  and  $\xi$  reads  $\epsilon_F(\alpha) = \xi^{1/3}$ , where for the case of a coherent state the equality  $\epsilon_F = |\alpha|^2$  must be substituted.

We are ready to present the main result of this paper, a limit of maximum entropy for the atomic density operator of the damped Jaynes–Cummings model. The limit is given by

$$\lim_{\substack{\kappa \rightarrow 0 \\ \gamma \rightarrow 0 \\ t \rightarrow \infty \\ \xi \rightarrow \infty}} \tilde{\rho}_A(t) = \frac{1}{2} \mathbf{1}_2. \tag{75}$$

The tilde denotes that the constraints  $\kappa = \kappa'/\xi, \gamma = \gamma'/\xi, t = t'\xi$  and  $\epsilon_F = \xi^{1/3}$  must be obeyed. The parameters  $\kappa', \gamma', t', \bar{n}, |\Delta|$  and  $\Gamma$  may have any fixed positive value, provided that condition (8) is respected. Finally, one should not forget that the initial density operator must be in keeping with property (68).

For the case of  $\Delta \neq 0$  we do not go through a detailed derivation of (75), but merely make a qualitative comment. An inspection of dependences shows that the detuning parameter



manifests itself in (55) through the ratio  $\Delta/n^{1/2}$ . Owing to (69), the photon number  $n$  becomes large in the limit (75). Therefore, the aforementioned ratio will tend to zero, and the limit will produce the same answer as for the case of zero detuning. The soundness of this argument can be verified immediately for the contributions to (56), which emanate from the  $l = 1$  and  $l = 2$  terms of (58). In contrast, an adequate treatment of all terms of (58) demands major efforts. Hence, for  $\Delta \neq 0$  the limit (75) should be regarded as a conjecture.

## 5. Extension to pure atomic damping

### 5.1. Bound on diagonal matrix elements

Until now we have taken the parameter for cavity damping  $\kappa$  as strictly positive. In the following, we shall study the atomic evolution for the case of  $\kappa = 0$  and  $\gamma > 0$ ; then all of the energy losses stem from spontaneous emission of photons by the two-level atom. Consequently, the damping mechanism gets a sequential character. The cavity mirrors can let through any number of photons at a time, but the atom can emit only one photon at a time.

By choosing  $\kappa$  equal to zero, we intend to demonstrate that a change of damping mechanism does not affect the existence of a limit of maximum entropy. In other words, in this section the important issue of universality is under investigation. For that reason we must maintain mathematical rigour. To avoid a high degree of complexity, we set  $\Delta$ ,  $\Gamma$  and  $d_\infty + 1/2$  equal to zero. We thus work at zero temperature. The atomic damping parameter  $\gamma$  will become small, so the inequality  $\gamma < 1$  can be freely used. Last, we opt for the initial condition

$$\rho(t = 0) = \rho_A \otimes |q\rangle\langle q| \quad (76)$$

where  $\rho_A$  stands for any atomic density matrix and  $|q\rangle$  denotes a number state, with  $q = 1, 2, 3, \dots$

From (10) and (11) it follows that the atomic density matrix is determined by

$$\rho_A(t)_{11} = \sum_{n=0}^{\infty} \mathbf{v}''(t; n, n)_1 \quad \rho_A(t)_{12} = \sum_{n=0}^{\infty} \mathbf{v}''(t; n, n-1)_3. \quad (77)$$

For the evaluation of the above sums we return to (12), where the choice  $\kappa = \lambda = \Delta = \Gamma = 0$  is made. Upon performing Laplace transformation and iteration [14], we arrive at

$$\begin{aligned} \rho_A(t)_{11} = & \rho_{A,11}[2f_1(t; 0, q) + 4\gamma^2 f_1(t; 1, q-1) + 16q\gamma^2 f_1(t; 2, q-2) + f_2(t; 1, q)] \\ & + \rho_{A,22}[2f_1(t; 0, q-1) + f_2(t; 0, q-1)]. \end{aligned} \quad (78)$$

The new functions are given by

$$f_1(t; h, q) = \sum_{k=0}^q \frac{(q+1)!(-1)^h \gamma^k e^{-\gamma t}}{k!(q-k)!4^{h+1}} \sum_{l=0}^k \binom{k}{l} t^{k-l} g(x=0; k+h, l, q+h+1) \quad (79)$$

and

$$\begin{aligned} f_2(t; h, q) = & \frac{1}{4} \sum_{\eta=\pm 1} \sum_{k=0}^q \sum_{l=0}^k \frac{q!(-1)^l}{l!(q-k)!(k-l)!} \left[ \frac{i\eta\gamma}{2u(q-l)} \right]^k \\ & \times \left[ 2h + \frac{i\eta\gamma}{u(q-l)} - \frac{q+1}{u^2(q-l)} \right] \exp[-\gamma t + 2i\eta t u(q-l)] \end{aligned} \quad (80)$$

with  $h = 0, 1, 2$  and  $q + 1$  non-negative. We have to work out the derivative

$$g(x; k, l, q) = \left(\frac{d}{dx}\right)^l \prod_{s=0}^k (x^2/4 - \gamma^2/4 + q - s)^{-1} \tag{81}$$

for  $1 \leq l \leq k$  and set  $x = 0$  afterwards.

In (79) we put the summation index  $l$  equal to zero; we call the ensuing function  $f_3(t; h, q)$ . For  $l = 0$  a differentiation need not be performed in (81), so we obtain after some minor algebra

$$f_3(t; h, q) = \frac{(q + 1)!(-1)^h e^{-\gamma t}}{(q + h + 1)!4^{h+1}} \sum_{k=0}^q \frac{(\gamma t)^k}{k!} + \gamma^2 \sum_{k=0}^q \sum_{l=-h}^k \frac{(q + 1)!(q - l)!(-1)^h (\gamma t)^k e^{-\gamma t}}{k!(q + h + 1)!(q - k)!4^{h+2}} \prod_{s=q-k+1}^{q-l+1} (s - \gamma^2/4)^{-1}. \tag{82}$$

The first term on the right-hand side corresponds to the choice of  $\gamma = 0$  in (81).

The case of  $l \geq 1$  can be dealt with on the basis of the following auxiliary identities:

$$\left(\frac{d}{dx}\right)^l f(x^2) = \sum_{p=0}^{\lfloor l/2 \rfloor} \frac{l!(2x)^{l-2p}}{p!(l - 2p)!} \left(\frac{d}{dx^2}\right)^{l-p} f(x^2) \tag{83}$$

and

$$\left(\frac{d}{dx}\right)^l \prod_{s=0}^k (x - s)^{-1} = (-1)^l l! \prod_{s=0}^k (x - s)^{-1} \sum_{r=1}^{\min(l, k+1)} \sum_{s_1, s_2, \dots, s_r=0}^k \frac{(x - s_1)^{r-l-1}}{(r - 1)! \prod_{j=2}^r (s_1 - s_j)}. \tag{84}$$

The prime imposes the condition that all indices  $\{s_j\}$  be different from each other. Verification of (83) and (84) can be done with the help of induction in  $l$ .

In (79) we exclude the term with  $l = 0$ ; we call the ensuing function  $f_4(t; h, q)$ . In explicit terms, one has  $f_1 = f_3 + f_4$ . The new function can be expressed as

$$f_4(t; h, q) = \sum_{k=2}^q \sum_{p=1}^{\lfloor k/2 \rfloor} \sum_{r=1}^{\min(p, k+h+1)} \sum_{s_1, s_2, \dots, s_r=0}^{k+h} \frac{(q + 1)!(-1)(-4)^{-h-p-1} \gamma^k t^{k-2p} e^{-\gamma t} (q + h + 1 - \gamma^2/4 - s_1)^{r-p-1}}{(k - 2p)!(q - k)!(r - 1)! \prod_{j=2}^r (s_1 - s_j) \prod_{s=0}^{k+h} (q + h + 1 - \gamma^2/4 - s)}. \tag{85}$$

With the notation  $[\dots]$  the entire function is meant. The results (80), (82) and (85) completely specify the diagonals of the atomic density matrix. Our next task is the construction of suitable bounds.

To establish a bound on the double summation in (82), we take the norm of the summand. In the product over  $s$  we set  $\gamma$  equal to 2 for  $s \geq q - k + 2$ , thus reducing the product to a factorial. In the sum over  $k$  we separate the term with  $k = q$  from the others, for which we choose  $\gamma = 2$  again. These manipulations lead us to the inequality

$$\left| f_3(t; h, q) + (-4)^{-h-1} \prod_{s=2}^{h+1} (q + s)^{-1} \right| \leq \frac{(\gamma t)^q}{q!} + \frac{2\gamma^2(1 + \gamma t)(h + 1)}{1 - \gamma^2/4}. \tag{86}$$

The second term on the left-hand side and the first term on the right-hand side arise from the summation over  $k$  in (82).

The derivation of a bound on  $f_4$  starts from the inequality

$$\frac{(q+h+1)!}{(q-k)!} \prod_{s=0}^{k+h} (q+h+1-\gamma^2/4-s)^{-1} \leq (1-\gamma^2/4)^{-q-h-1} \tag{87}$$

which is valid for  $k \leq q$ . Next, we separate the sum over  $s_1$  from the sums over  $\{s_j\}_{j=2}^r$ , and omit the prime. This offers the possibility of employing the inequality

$$\sum_{s=0(s \neq s_1)}^k |s-s_1|^{-1} \leq 2 \sum_{s=1}^k s^{-1}. \tag{88}$$

The right-hand side appears  $r-1$  times.

The sum over  $s_1$  can be controlled by retaining the largest term, which is the one with  $s_1 = k+h$ , and multiplying by the number of terms, which amounts to  $k+h+1$ . These operations provide us with the intermediate inequality

$$|f_4(t; h, q)| \leq \sum_{k=2}^q \sum_{p=1}^{\lfloor k/2 \rfloor} \sum_{r=1}^p \frac{(k+h+1)(1-\gamma^2/4)^{-q-h-1} \gamma^k t^{k-2p} e^{-\gamma t}}{4^{h+p+1} (k-2p)! (r-1)! \prod_{s=1}^h (q+s+1)} \times (q-k+1-\gamma^2/4)^{r-p-1} \left( 2 \sum_{s=1}^{k+h} s^{-1} \right)^{r-1}. \tag{89}$$

In the factors of  $(k+h+1)$  and  $(q-k+1-\gamma^2/4)^{r-p-1}$ , as well as in the sum over  $s$ ,  $k$  may be set equal to its maximum value of  $q$ . Next, we perform the transformation  $k' = k-2p$ , and interchange the sums over  $p$  and  $r$ . Our last step consists of the transformation  $p' = p-r$ .

Now we are in a position to replace the upper boundaries of all three summations by infinity. This results in two exponential series and one geometric series. Further simplification is brought about by the inequality  $\sum_{s=1}^q s^{-1} \leq 2q^{1/2}$ . In final form, the bound is given by

$$|f_4(t; h, q)| \leq 3\gamma^2 q (1-\gamma^2/4)^{-q} \exp(2\gamma^2 q^{1/2}). \tag{90}$$

We have used that  $h$  is smaller than 2.

The function  $f_2$  does not pose any particular problem. Expression (80) gives rise to the bound

$$|f_2(t; h, q)| \leq 14q(1+2\gamma)^q e^{-\gamma t}. \tag{91}$$

We have utilized the fact that for  $\gamma \leq 1$  the result  $u(n) \geq 1/2$  is true.

### 5.2. Bound on off-diagonal matrix elements

The evaluation of the off-diagonal (77) does not differ much from the procedure that determines the diagonal elements. By combining Laplace transformation with iterative methods, one obtains

$$\rho_A(t)_{12} = \frac{1}{16} \rho_{A,12} [2f_5(t; q) + f_6(t; q) + 8f_7(t; q) + f_8(t; q)]. \tag{92}$$

If we introduce the abbreviations

$$r_{\eta_1, \eta_2}(n) = \eta_1 u(n) + \eta_2 u(n-1) + i\gamma \quad s_\eta = \eta u(0) + i\gamma/2 \tag{93}$$

then the functions on the right-hand side of (92) can be expressed as

$$f_5(t; q) = \sum_{\eta_1, \eta_2 = \pm 1} \frac{\eta_1 \eta_2 \exp[i t r_{\eta_1, \eta_2}(q)]}{u(q) u(q-1) r'_{\eta_1, \eta_2}(q)} \left[ r'_{\eta_1, \eta_2}(q) |r_{\eta_1, \eta_2}(q)|^2 - 2q r'_{\eta_1, \eta_2}(q) - r_{\eta_1, \eta_2}(q) \right]$$

$$\begin{aligned}
 f_6(t; q) &= \sum_{\eta_1, \eta_2 = \pm 1} \sum_{k=1}^{q-1} \sum_{l=0}^k \frac{\eta_1 \eta_2 \exp [i t r_{\eta_1, \eta_2}(q-l)] q!(4i\gamma)^k}{u(q-l) u(q-l-1)(q-k)!} [r'_{\eta_1, \eta_2}(q-l)]^{k-2} \\
 &\quad \times [r'_{\eta_1, \eta_2}(q-l) r_{\eta_1, \eta_2}^*(q-l) - 1] [r'_{\eta_1, \eta_2}(q-l) r_{\eta_1, \eta_2}(q-l) + 1] \\
 &\quad \times \prod_{s=0(s \neq l)}^k \prod_{\eta_3, \eta_4 = \pm 1} [r'_{\eta_3, \eta_4}(q-s) - r'_{\eta_1, \eta_2}(q-l)]^{-1} \\
 f_7(t; q) &= \sum_{\eta = \pm 1} \eta \exp [i t s_\eta] \frac{q!(4i\gamma)^q}{u(0)} (s_\eta^*)^{q-1} \prod_{s=1}^q \prod_{\eta_1, \eta_2 = \pm 1} [r'_{\eta_1, \eta_2}(s) - s_\eta^*]^{-1} \\
 f_8(t; q) &= \sum_{\eta_1, \eta_2 = \pm 1} \sum_{k=1}^q \frac{\eta_1 \eta_2 \exp [i t r_{\eta_1, \eta_2}(k)] q!(4i\gamma)^q}{u(k) u(k-1)} [r'_{\eta_1, \eta_2}(k)]^{q-2} [r'_{\eta_1, \eta_2}(k) r_{\eta_1, \eta_2}(k) + 1] \\
 &\quad \times \prod_{\eta = \pm 1} [r'_{\eta_1, \eta_2}(k) - s_\eta^*]^{-1} \prod_{s=1(s \neq k)}^q \prod_{\eta_3, \eta_4 = \pm 1} [r'_{\eta_3, \eta_4}(s) - r'_{\eta_1, \eta_2}(k)]^{-1}. \tag{94}
 \end{aligned}$$

The notation  $r' = \text{Re}(r)$  has been used.

The inequalities  $\gamma \leq 1$  and  $1/2 \leq u(q) \leq (q+1)^{1/2}$  lead us to the bound

$$|f_5(t; q)| \leq 10^4 q^2 e^{-\gamma t}. \tag{95}$$

Moving on to  $f_6$ , we observe that the denominators in the product over  $s$  can be controlled by means of the inequalities

$$\prod_{\eta_1, \eta_2 = \pm 1} |r'_{\eta_1, \eta_2}(q-s) + r'_{\eta_2, \eta_1}(q-l)|^{-1} \leq \frac{\delta_{\eta, +1}}{|l-s| u^2(q-l)} + \frac{4\delta_{\eta, -1} [r'_{+,+}(q-l)]^2}{|l-s|} \tag{96}$$

where  $\eta$  equals  $\pm 1$  and  $l$  differs from  $s$ . The choice of  $\eta = +1$  ( $-1$ ) corresponds to the terms of  $f_6$  with  $\eta_1 \eta_2 = +1$  ( $-1$ ). For the latter case, it is important to take advantage of the identity

$$r'_{+,+}(q) r'_{+,-}(q) = 1. \tag{97}$$

The bound for  $\eta = +1$  is much smaller than the one for  $\eta = -1$ , so the former may be omitted.

One can finalize the estimate for  $f_6$  as follows:

$$|f_6(t; q)| \leq 10^5 q^3 (1 + 128\gamma q^{1/2})^q e^{-\gamma t}. \tag{98}$$

We also need the bound

$$|f_7(t; q)| + |f_8(t; q)| \leq 10(8\gamma)^q e^{-\gamma t/2} + 10^2 q^2 (128\gamma q^{1/2})^q e^{-\gamma t}. \tag{99}$$

It can be proved in the same manner as (95) and (98).

### 5.3. Limit of maximum entropy

All of the bounds, which have been proved in the preceding subsections, converge to zero if  $\gamma$  is made small such that the products  $\gamma^{4/3} t$  and  $\gamma^{2/3} q$  remain constant. As a consequence, all functions  $\{f_j\}_{j=2}^8$  converge to zero as well, except for  $f_3(t; h = 0, q)$ . This last function tends to  $1/4$ .

With the help of the normalization condition  $\rho_{A,11} + \rho_{A,22} = 1$ , we deduce from (78) and (92) the following statement:

$$\lim_{\substack{\gamma \rightarrow 0 \\ t \rightarrow \infty \\ q \rightarrow \infty}} \hat{\rho}_A(t) = \frac{1}{2} \mathbf{1}_2. \tag{100}$$

The hat indicates that the limit must be taken under the constraint that  $\gamma^{4/3}t$  and  $\gamma^{2/3}q$  are constant. The parameters  $\kappa$ ,  $\Delta$  and  $\Gamma$  are identical to zero. The same goes for the temperature of the atomic reservoir. We emphasize that the route from the master equation (3) to the above limit of maximum entropy is completely rigorous.

The limit (100) may be regarded as a counterpart of (75). We thus conclude that a modification of the damping mechanism does not affect the status of the central state as a stable fixed point. Yet there is a difference with the case of photonic damping. If the cavity mirrors act as energy sinks, then the convergence to the central state is driven by an exponential factor of  $\exp(-\kappa tq)$ . The number of initial photons  $q$  becomes large, so the product  $\kappa t$  may be chosen as finite. For the case of pure atomic damping the photon number  $q$  is lacking. The bare exponential  $\exp(-\gamma t)$  is responsible for convergence. Hence, the product  $\gamma t$  can no longer stay finite. This is the mathematical consequence of adopting a damping mechanism that has a sequential character.

## 6. Conclusion

One of the main objectives of non-equilibrium statistical mechanics is the microscopic derivation of a general principle that is akin to the second law of thermodynamics [4]. This almost antique problem [15] cannot be ignored if we wish to complete our understanding of irreversible behaviour. We should systematically explore what the dynamics of open quantum systems can tell us about entropy and related notions.

We can evade the unyielding many-body theories by focusing on the evolution of a small quantum system that is coupled to a large thermal reservoir. The advantage is that quite often, it is possible to obtain the exact solution for the density operator. Then we can try to acquire information on the dynamics without the use of numerical means. The choice to banish the origin of irreversibility to the reservoir furthermore enables us to work in a Hilbert space of finite dimension. Such a setting minimizes the risk of being hindered by mathematical technicalities.

The relevance of quantum entropy for small open systems cannot be questioned. Many encouraging results were reported already [16]. For instance, by employing the fruitful concept of relative entropy [17], and assuming a Markovian master equation for the density operator [18], one succeeds in formulating a quantum counterpart of the second law [12, 19]. The associated entropy production is strictly non-negative. This entropy production derives, at root, from the unitary and reversible quantum dynamics of system and reservoir together. Hence, a sound connection to the Schrödinger equation definitely exists.

In carrying out a microscopic derivation of Markovian master equations, one cannot avoid taking either a weak-coupling [10] or a singular-coupling [20] limit. As desired, these procedures bring about irreversibility, but on the other hand, they erase all traces of unitary dynamics. Consequently, in the evolution of the quantum system an essential stage is lacking, namely, the cross-over from quasi-reversible dynamics to smooth irreversible decay. Precisely this stage should provide us with new and valuable insights into the onset of irreversibility.

In our opinion, a quantum substitute for the second law may be called mature, only if it can be applied outside the framework of Markovian master equations. Unfortunately, it is very hard to get access to the non-Markovian regime, as every worker on quantum dissipation knows. One may make a start by undertaking some careful case studies, and deriving as many exact results as possible. The damped Jaynes–Cummings model is perfectly suited for these purposes. Evidently, the full density operator for the atom and field mode obeys a Markovian master equation. However, the atomic density matrix, which is obtained by taking a partial trace over the field, undergoes a highly non-Markovian evolution. For times of order  $g^{-1}$ ,

where  $g$  denotes the coupling constant of the Jaynes–Cummings interaction, the dynamics exhibits the well-known Rabi oscillations [1]. These indeed possess a quasi-reversible character.

In paper I, an extensive numerical study of the Jaynes–Cummings model with cavity damping was presented. The evolution of the atomic density matrix was scrutinized for the case of weak damping. The atom conformed to expectations by making a very slow transition to the final stage of exponential decay. Surprisingly, the central state  $\mathbf{1}_2/2$  turned out to be the main attractor in atomic phase space. For small damping parameter, typically  $\kappa \leq 0.01$ , and high energy density of the initial electromagnetic field, typically  $\epsilon_F \geq 25$ , all atomic trajectories met at the central state. There they stayed for a long time of the order of  $(g\kappa)^{-1}$ , whereafter they all followed the same Markovian path to the atomic ground state. Of course, one should not overlook the possibility, however improbable it may be, that the aforescribed course of events was the result of artefacts, caused by an unrealistic choice of parameters. For instance, in paper I the cavity temperature was set equal to zero. If that argument were valid, then there would be no point in searching for any physically relevant conclusions.

In this paper, we have derived a universal limit of maximum entropy for the two-level atom. With this result in hand, we can brush aside all reservations about the role of the central state as the main attractor. Indeed, our treatment is completely free from any particular choice of parameters or states. The temperature and detuning differ from zero; both the atom and field are coupled to a thermal reservoir, albeit that equation (8) must be satisfied; any state that complies with the constraint (68) may be taken as the initial density operator. The last possibility implies that entanglement of the atom and field may occur at time zero. In addition, we have checked that the restriction to pure atomic damping does not harm the limit of maximum entropy. Therefore, we believe that our findings might be of help in clarifying the status of entropy for evolutions of non-Markovian nature, far from thermodynamic equilibrium. Proposals for further research were already made in I. As a start, one could try to devise a formula for entropy production that properly deals with the oscillatory character of limit (75).

In closing, we point out once more that, due to identity (8), the composite system of atom and field mode respects the condition of detailed balance. Hence, it is possible to reconcile the asymptotic evolution of the composite system with the principles of irreversible thermodynamics. Use should be made of the powerful formalism that was developed for dissipative quantum evolutions of Markovian character [12, 18, 19]. One proves that the density operator  $\rho(t)$  of the composite system converges to the factorized thermal state (9) as  $t$  becomes large. Furthermore, one can harmonize the accompanying irreversible behaviour with the second law of thermodynamics. Given these satisfactory achievements, one might reject any attempt to search for a thermodynamic explanation of the asymptotic evolution of the atom alone.

The foregoing point of view would be viable, if the atomic evolution were not displaying any signs of irreversible behaviour. This is certainly the case for a single gas atom that is contained in a vessel together with a large number of other gas atoms. However, in the damped Jaynes–Cummings model the atom interacts with a large thermal reservoir. Irreversible behaviour is manifestly present and, more important still, it is of a novel type. In atomic phase space all trajectories convene at the central state. Hence, the attractor is not the thermal but the central state. Moreover, the approach to the attractor is not smooth but oscillatory. It will be most instructive to investigate how the irreversible atomic evolution, as found in this work, relates to the principles of non-equilibrium thermodynamics. The damped Jaynes–Cummings model represents a unique chance to improve our understanding of quantum irreversibility.

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